

Teleparallel Gravity in Five Dimensional Theories

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(Dated: July 22, 2014)

Abstract

We study teleparallel gravity in five-dimensional spacetime with particular discussions on Kaluza-Klein (KK) and braneworld theories. We directly perform the dimensional reduction by differential forms. In the braneworld theory, the teleparallel gravity formalism in the Friedmann-Lemaître-Robertson-Walker cosmology is equivalent to GR due to the same Friedmann equation, whereas in the KK case the reduction of our formulation does not recover the effect as GR of 4-dimensional spacetime.

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I. INTRODUCTION

Extra dimension theory was first studied by Kaluza [1] and Klein [2], the so-called KK theory, in order to unify electromagnetism and gravity by gauge theory. Electromagnetic fields come from the extra 5th-dimension which is compactified in some small scale. It is usually used to explain the hierarchy problems with the effective Planck scale in 4-dimension by the dimensional reduction. The *large extra dimension* was proposed by Arkani-Hamed, Dimopoulos and Dvali [3], referred to the *braneworld* theory. The theory with a brane as the solitonic solution for a physical object is inspired from supergravity as well as superstring theory [4]. The ordinary matter fields are localized on the *brane* embedded into a spacetime of a higher dimension called *bulk*. Randall and Sundrum [5] gave two braneworld models based on particular *non-factorizable* metrics, named *RS-I* and *II* models, leading to a *warp extra dimension* between two 3-branes to solve the hierarchy problem and an compactification to generate 4-dimensional gravity, respectively.

On the other hand, *teleparallel gravity*, which an alternative gravity theory other than GR, was first considered by Einstein [6] in terms of *absolutely parallelism*. The Lagrangian of *teleparallel equivalent to general relativity* (TEGR) is referred to as the torsion scalar T . Recently, several types of gravity theories with T , such as the teleparallel dark energy [7] and $f(T)$ [8] models, have been used to explain the acceleration of the universe.

Extra dimension theories in teleparallel gravity have been explored in the literature [9–13]. In this article, we first set up a general geometrical scenario of TEGR and then investigate the general behavior of the bulk as well as the projected effect on the brane. In the calculations, we keep our geometric construction as general as possible so that it is easy to compare our results with those in [9–13]. In particular, we concentrate on the KK (without vector fields) and the braneworld theories. The application of the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology in the braneworld scenario is also discussed.

II. TELEPARALLEL GRAVITY IN FIVE-DIMENSIONAL SPACETIME

We formulate teleparallel gravity in high dimensions. Our geometrical settings are given as follows. Let (M, g) be a 4-dimensional spacetime (hypersurface) isometrically embedded into a 5-dimensional spacetime (the bulk) (V, \bar{g}) by $f : M \rightarrow V$. We use the convention

$dx^M = (dx^\mu, dx^5)$ for the coordinate dual basis of V with capital Latin letters $M, N = 0, 1, 2, 3, 5$ and Greek letters $\mu, \nu, \dots = 0, 1, 2, 3$ as the coordinate indices of M , and middle Latin letters $i, j, k, \dots = 1, 2, 3$ as the spatial indices of M . An orthonormal frame (tetrad) $\bar{\vartheta}^A = (\bar{\vartheta}^a, \bar{\vartheta}^5)$ for V is indexed by capital Latin letters $A, B, C, \dots = 0, 1, 2, 3, 5$ with Latin letters $a, b, c, \dots = 0, 1, 2, 3$ for tetrads on M , while the middle Latin letters $i, j, k, \dots = 1, 2, 3$ share with spatial coordinate indices¹.

For a tetrad (e_0, \dots, e_3) on M , it can be naturally extended as a tetrad $(\bar{e}_0, \dots, \bar{e}_3, \bar{e}_5)$ on V , i.e. $\bar{e}_a := f_*(e_a)$, where \bar{e}_5 is the unit normal vector field to M . The corresponding coframes are $(\vartheta^0, \dots, \vartheta^3)$ for M and $(\bar{\vartheta}^0, \dots, \bar{\vartheta}^3, \bar{\vartheta}^5)$ for V with $f^*(\bar{\vartheta}^a) = \vartheta^a$. We shall identify M with $\bar{M} := f(M) \subset V$, and $\vartheta^a \in T^*M$ with $\bar{\vartheta}^a \in T^*\bar{M}$..., etc., interchangeably. Quantities with bars, e.g. \bar{e}_A , represent objects viewed in V . The metric signature is fixed as $(-, +, +, +, \varepsilon)$ and the sign of the 5th-dimension is denoted by $\varepsilon := \bar{g}(\bar{e}_5, \bar{e}_5) = \pm 1$.²

In the 4-dimensional teleparallel theory, one uses a tetrad ϑ^a to formulate the gravitational theory. The metric g of M is given by

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} \vartheta^a \otimes \vartheta^b, \quad (2.1)$$

where η_{ab} is the Minkowski metric. Occasionally, one writes the tetrad in terms of the local coordinate, such as $e_a = e_a^\mu \partial_\mu$ and $\vartheta^a = e_a^\mu dx^\mu$. With a given tetrad e_a on M , we can define the Weitzenböck connection by

$$\nabla_{e_a}^W e_b \equiv 0, \quad (\text{for all } a, b) \quad (2.2)$$

and the Weitzenböck connection 1-form ω_a^b on M . It is easy to observe that the Weitzenböck connection (2.2) yields a vanishing curvature since

$$R^d{}_{cab} e_d = \nabla_{e_a}^W \nabla_{e_b}^W e_c - \nabla_{e_b}^W \nabla_{e_a}^W e_c - \nabla_{[e_a, e_b]}^W e_c \equiv 0. \quad (2.3)$$

However, the connection takes torsion to manifest properties of spacetime and the gravitational effects. The torsion 2-form on M is given by $T^a = \nabla^W \vartheta^a = \frac{1}{2} T^a{}_{bc} \vartheta^b \wedge \vartheta^c$ with the torsion components $T^a{}_{bc} = e_b(e_c^a) e_c^\mu - e_c(e_b^a) e_b^\mu$. Consequently, the torsion scalar on M is given by

$$T = \frac{1}{4} T_{abc} T^{abc} + \frac{1}{2} T_{abc} T^{cba} - T^b{}_{ba} T^c{}^a{}_c \quad (2.4a)$$

$$= \frac{1}{4} T_{\mu\nu\sigma} T^{\mu\nu\sigma} + \frac{1}{2} T_{\mu\nu\sigma} T^{\sigma\nu\mu} - T^\nu{}_{\nu\mu} T^\sigma{}^\mu{}_\sigma. \quad (2.4b)$$

¹ These spatial indices should cause no confusion as it can be easily read off from the context.

² Note that $1/\varepsilon = \varepsilon = \pm 1$ is used in the calculation.

In the following discussions, we adopt differential forms to reduce large tensor calculations. Based on differential forms, the torsion scalar (2.4a) can be rewritten as a 4-form [14, 15]

$$\mathcal{T} = T_a \wedge \star \left[{}^{(1)}T^a - 2 {}^{(2)}T^a - \frac{1}{2} {}^{(3)}T^a \right] := -\frac{1}{2} T_a \wedge H^a, \quad (2.5)$$

where

$$\begin{aligned} {}^{(1)}T^a &:= T^a - {}^{(2)}T^a - {}^{(3)}T^a, \\ {}^{(2)}T^a &:= \frac{1}{3} \vartheta^a \wedge i_{e_b}(T^b), \\ {}^{(3)}T^a &:= \frac{1}{3} i_{e^a}(\vartheta_b \wedge T^b), \\ H^a &:= (-2) \star \left[{}^{(1)}T^a - 2 {}^{(2)}T^a - \frac{1}{2} {}^{(3)}T^a \right] \end{aligned} \quad (2.6)$$

with \star the Hodge dual operator in g of M and $i_v(\omega)$ the interior product of v with a k -form ω . Similarly, the 5-form torsion scalar $\bar{\mathcal{T}}$ of V is defined as (2.5), namely

$$\bar{\mathcal{T}} = \bar{T}_A \wedge \bar{\star} \left[{}^{(1)}\bar{T}^A - 2 {}^{(2)}\bar{T}^A - \frac{1}{2} {}^{(3)}\bar{T}^A \right], \quad (2.7)$$

where $\bar{\star}$ is the Hodge dual operator in (V, \bar{g}) and $\bar{T}^A := \nabla^W \bar{\vartheta}^A = \bar{d}\bar{\vartheta}^A + \bar{\omega}_B^A \wedge \bar{\vartheta}^B = \bar{d}\bar{\vartheta}^A$ is the torsion 2-form on V , in which the two kinds of differentials $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ and $\bar{d} : \Omega^k(V) \rightarrow \Omega^{k+1}(V)$ should be carefully distinguished, along with the requirement $\bar{d}|_M = d$. The gravitational action on V is given by

$${}^{(5)}S = \int -\frac{\bar{\mathcal{T}}}{2\kappa_5} = \int -\frac{\bar{T}}{2\kappa_5} d\text{vol}^5, \quad (2.8)$$

where $\kappa_5 = 8\pi G^{(5)}$ represents the 5-dimensional gravitational coupling, \bar{T} stands for the torsion scalar of V , and $d\text{vol}^5 = {}^{(5)}e d^5x = \det(e_M^A) d^5x$ is the volume form of V .

Since the hypersurface M is at least an immersion of V , there always exists a coordinate system such that we can write the 5D metric \bar{g} as the form

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu}(x^\mu, y) & 0 \\ 0 & \varepsilon \phi^2(x^\mu, y) \end{pmatrix} \quad (2.9)$$

where $y = x^5$. Within such a coordinate, we have a preferred frame for V with

$$\bar{e}_A = \left(e_a, \frac{1}{\phi} \frac{\partial}{\partial y} \right), \quad \bar{\vartheta}^A = (\bar{\vartheta}^a, \phi dy), \quad (2.10)$$

which will be used in this study unless particularly specified.

III. EFFECTIVE GRAVITATIONAL ACTION ON THE HYPERSURFACE M

The usual high dimensional theory in GR, such as KK and braneworld scenarios [16, 17], uses the *Gauss-Codazzi equation* to relate the equation of motions between the hypersurface (M, g, ∇) and the bulk $(V, \bar{g}, \bar{\nabla})$. The crucial ingredient that guides the physical connection is through the *extrinsic curvature*, defined by

$$K(X, Y) := -\varepsilon \bar{g} \left(\bar{\nabla}_{f_*(X)} \bar{e}_5, f_*(Y) \right) \quad (X, Y \in \mathfrak{X}(M)), \quad (3.1)$$

However, in TEGR, since the Weitzenböck connection implies $K_{ab} = 0$ by (2.2) and the vanishing curvature by (2.3), the Gauss-Codazzi equation

$$\bar{R}^a{}_{bcd} = R^a{}_{bcd} - \varepsilon K_c^a K_{db} + \varepsilon K_d^a K_{cb} \quad (3.2)$$

is simply a zero identity, so that the hypersurface is like a flat-paper in an Euclidean space \mathbb{R}^3 . In fact, we can construct an *extrinsic torsion* [18, 19] similar to the extrinsic curvature of GR in (3.1) to describe the dynamics of the embedded spacetime in TEGR, which is given by:

$$\begin{aligned} B(X, Y) &:= \varepsilon \bar{g} \left(\bar{T}(f_*(X), f_*(Y)), \bar{e}_5 \right), \quad (X, Y \in \mathfrak{X}(M)), \\ &= \varepsilon \left[\bar{g} \left(f_*(X), \bar{\nabla}_{f_*(Y)} \bar{e}_5 \right) - \bar{g} \left(f_*(Y), \bar{\nabla}_{f_*(X)} \bar{e}_5 \right) \right]. \end{aligned} \quad (3.3)$$

However, since we use tetrads $(\vartheta^0, \dots, \vartheta^3)$ and $(\bar{\vartheta}^0, \dots, \bar{\vartheta}^3, \bar{\vartheta}^5)$ with the Weitzenböck connections of M and V in Sec. II, the extrinsic torsion (3.3) does not give us more information from the extra-dimension due to

$$B_{ab} = \varepsilon \bar{g} \left(\bar{e}_a, \bar{\nabla}_{\bar{e}_b} \bar{e}_5 \right) - \varepsilon \bar{g} \left(\bar{e}_b, \bar{\nabla}_{\bar{e}_a} \bar{e}_5 \right) = 0, \quad (\text{for all } a, b). \quad (3.4)$$

The extra degree of freedom in TEGR actually is contained in a torsion 2-form. With the general setting above, we can now calculate the projected effect onto M from V . We decompose the torsion \bar{T}^a of V into normal and parallel components respect to M by

$$\bar{T}^a = T^a + \bar{T}^a{}_{b5} \bar{\vartheta}^b \wedge \bar{\vartheta}^5, \quad (3.5)$$

where $T^a = \frac{1}{2} T^a{}_{bc} \vartheta^b \wedge \vartheta^c$ is the torsion 2-form on M . In the frame (2.10), the nonvanishing torsion components of V are $T^a{}_{bc}$, $\bar{T}^a{}_{5b} = e_5(e_b^a) e_b^\mu$ and $\bar{T}^5{}_{b5} = \frac{1}{\phi} e_b(\phi)$. In particular, if we let the ambient space V be a local product of $U \times W$, where $U \subseteq M$ is an open in M and

W corresponds to the extra spatial dimension. The local product structure of V allows us to integrate over the base space U of M ,

$$S_{\text{bulk}} = \frac{-1}{2\kappa_5} \int_U \int_W \left(T + \frac{1}{2} (T_{ab5} T^{ab5} + T_{a5b} T^{b5a}) + \frac{2}{\phi} e_a(\phi) t^a - t_5 \cdot t^5 \right) \phi dy d\text{vol}^4, \quad (3.6)$$

where T is the (induced) 4-dimensional torsion scalar defined in (2.4). The equation in (3.6) provides us with the general effective action for the hypersurface M in TEGR. In the next, we concentrate on two specific theories of braneworld and Kaluza-Klein scenarios.

A. Braneworld Scenario

In the braneworld scenario, we set the hypersurface M located at $y = 0$ as a brane and specifying $V = M \times \mathbb{R}$ as a product manifold. From (3.6), the action on the bulk reads

$$S_{\text{bulk}} = \frac{-1}{2\kappa_5} \int_M \int_{\mathbb{R}} \left\{ \phi T + \phi \left(\frac{1}{2} (T_{ab5} T^{ab5} + T_{a5b} T^{b5a}) + \frac{2}{\phi} e_a(\phi) t^a - t_5 t^5 \right) \right\} dy d\text{vol}^4 \quad (3.7)$$

The first term of the parentheses in (3.7), recognized as $\int_M \int_{\mathbb{R}} \phi T \sqrt{-g} dy d^4x$, is the usual TEGR Lagrangian with a nonminimal coupled scalar field ϕ on the brane localized in the 5th-dimension, which is equivalent to the nonminimal coupled Hilbert action $\int_M \int_{\mathbb{R}} \phi R \sqrt{-g} dy d^4x$ of 4-dimension in GR. The second term arises from the 5th-dimensional component.

According to the *induced-matter theory*, the 5th-dimensional component and the flow along the 5th-dimension of the second term in (3.7) can be regarded as the induced-matter from *geometry*. It is the projected effect due to the extra spatial dimension. We note that the *mathematically equivalent* formulations between the induced-matter and braneworld theories have been demonstrated by Ponce de Leon in [20]. In Sec. IV, we shall discuss 5-dimensional field equations and the corresponding braneworld cosmology.

B. Kaluza-Klein Theory

In the KK theory, we take the space V locally as $U \times S^1$ and consider the 4-dimensional effective low-energy theory to obtain the *KK ansatz* in TEGR, that is

$$e_5(g_{\mu\nu}) = 0 \quad \text{or} \quad \frac{\partial}{\partial y} g_{\mu\nu} = 0 \quad (3.8)$$

with only the *massless Fourier mode* [21]. The metric is reduced to

$$\bar{g}_{MN} = \begin{pmatrix} g_{\mu\nu}(x^\mu) & 0 \\ 0 & \phi^2(x^\mu) \end{pmatrix} \quad (3.9)$$

with $\varepsilon = +1$. Due to the *KK ansatz*, we have $T^a_{b5} = 0$ and $t^5 = 0$ so that the extra-dimensional integration is trivial, *i.e.* $\int_{S^1} \phi(x^\mu) dy = 2\pi r \phi(x^\mu)$, where r is the radius of the 5th-dimension. As a result, we obtain

$$S_{\text{KK}} = \frac{-1}{2\kappa_4} \int_U (\phi T + 2 \partial_\mu \phi t^\mu) e d^4x, \quad (3.10)$$

where $\kappa_4 := \kappa_5/2\pi r$ is the *effective* 4-dimensional gravitational coupling constant. We point out that our result of (3.10) disagrees with that given in [12]. One can use a simple case with $F(T) = T$ in Eq. (5) of [12] to see that the resultant equation differs from ours in (3.10).

IV. FRIEMANN EQUATION OF BRANEWORLD SCENARIO IN TEGR

A. FLRW Brane Universe

We now study the teleparallel braneworld effect in cosmology. We assume that the brane M with $y = 0$ gives a homogeneous and isotropic universe. The bulk metric \bar{g} is maximally symmetric 3-space with spatially flat ($k = 0$), given by

$$\bar{g}_{MN} = \text{diag}(-1, a^2(t, y), a^2(t, y), a^2(t, y), \varepsilon \phi^2(t, y)) \quad (4.1)$$

by choosing a coframe with $\bar{\vartheta}^0 = dt$, $\bar{\vartheta}^i = a(t, y) dx^i$ and $\bar{\vartheta}^5 = \phi(t, y) dy$. Subsequently, the torsion 2-forms are

$$\bar{T}^0 = \bar{d}\bar{\vartheta}^0 = 0, \quad \bar{T}^i = \bar{d}\bar{\vartheta}^i = \frac{\dot{a}}{a} \bar{\vartheta}^0 \wedge \bar{\vartheta}^i + \frac{a'}{a\phi} \bar{\vartheta}^5 \wedge \bar{\vartheta}^i, \quad \bar{T}^5 = \frac{\dot{\phi}}{\phi} \bar{\vartheta}^0 \wedge \bar{\vartheta}^5, \quad (4.2)$$

where the *dot* and *prime* stand for the partial derivatives respect to t and y , respectively. The torsion scalar in (2.7) reads

$$\bar{\mathcal{T}} = \left[T + \left(\frac{3 - 9\varepsilon}{\phi^2} \frac{a'^2}{a^2} + 6 \frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi} \right) \right] d\text{vol}^5 \quad (4.3)$$

where $\varepsilon = +1$ and $T = 6\dot{a}^2/a^2$ is the usual 4-dimensional scalar torsion.

B. Equations of Motion

The gravitational field equations on the bulk can be derived from the formulation given in [14, 15]. The equations of motion on V are 4-forms

$$\bar{D}\bar{H}_A - \bar{E}_A = -2\kappa_5^{(5)}\bar{\Sigma}_A, \quad (4.4)$$

with

$$\begin{aligned} \bar{H}_A &= (-2)\bar{\star} \left({}^{(1)}\bar{T}_A - 2 {}^{(2)}\bar{T}_A - \frac{1}{2} {}^{(3)}\bar{T}_A \right), \\ \bar{E}_A &:= i_{\bar{e}_A}(\bar{\mathcal{T}}) + i_{\bar{e}_A}(\bar{T}^B) \wedge \bar{H}_B, \\ \bar{\Sigma}_A &:= \frac{\delta \bar{L}_{mat}}{\delta \bar{\vartheta}^A}, \end{aligned} \quad (4.5)$$

where $\bar{\Sigma}_A$ is the canonical energy-momentum 4-form of matter fields, and \bar{H}_A can be simplified as [15]

$$\bar{H}_A = (\bar{g}^{BC} \bar{K}_C^D) \wedge \bar{\star} (\bar{\vartheta}_A \wedge \bar{\vartheta}_B \wedge \bar{\vartheta}_D), \quad (4.6)$$

with $\bar{K}_C^D := \bar{\omega}_C^D - \tilde{\omega}_C^D$ being the contortion 1-form. Here, $\tilde{\omega}_C^D$ is the Levi-Civita connection 1-form with respect to the coframe with the unique expression of $\tilde{\omega}_C^D$, given by

$$\begin{aligned} \tilde{\omega}_i^0 &= \frac{\dot{a}}{a} \bar{\vartheta}^i, \quad \tilde{\omega}_0^i = \tilde{\omega}_i^0, \quad \tilde{\omega}_5^0 = \varepsilon \frac{\dot{\phi}}{\phi} \bar{\vartheta}^n, \quad \tilde{\omega}_0^5 = \varepsilon \tilde{\omega}_5^0, \\ \tilde{\omega}_j^5 &= -\varepsilon \frac{a'}{\phi a} \bar{\vartheta}^j, \quad \tilde{\omega}_5^j = -\varepsilon \tilde{\omega}_j^5, \quad \tilde{\omega}_j^i \equiv 0. \end{aligned} \quad (4.7)$$

From (4.6) and (4.7), we obtain the equations of motion of the bulk:

$$\begin{aligned} \bar{D}\bar{H}_0 - \bar{E}_0 &= 3 \left[\left(\frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi} \right) - \frac{\varepsilon}{\phi^2} \left(\frac{a''}{a} - \frac{a'}{a} \frac{\phi'}{\phi} \right) - \left(\frac{1+\varepsilon}{2\phi^2} \right) \frac{a'^2}{a^2} \right] \bar{\star} \bar{\vartheta}_0 \\ &\quad + \frac{3\varepsilon}{\phi} \left(\frac{\dot{a}'}{a} - \frac{a'}{a} \frac{\dot{\phi}}{\phi} \right) \bar{\star} \bar{\vartheta}_5 = -\kappa_5 \bar{\Sigma}_0, \\ \bar{D}\bar{H}_5 - \bar{E}_5 &= \frac{3}{\phi} \left(\frac{a'}{a} \frac{\dot{\phi}}{\phi} - \frac{\dot{a}'}{a} \right) \bar{\star} \bar{\vartheta}_0 + 3 \left[\left(\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} \right) - \left(\frac{1+\varepsilon}{2\phi^2} \right) \frac{a'^2}{a^2} \right] \bar{\star} \bar{\vartheta}_5 = -\kappa_5 \bar{\Sigma}_5. \end{aligned} \quad (4.8)$$

Note that the first equation in Eq. (4.8) is the Friedmann equation of the bulk. If we write $\bar{\Sigma}_A = \bar{T}_A^B \bar{\star} \bar{\vartheta}_B$, we get that

$$\left(\frac{\dot{a}^2}{a^2} + \frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi} \right) - \frac{1}{\phi^2} \left(\frac{a''}{a} - \frac{a'}{a} \frac{\phi'}{\phi} \right) - \frac{1}{\phi^2} \frac{a'^2}{a^2} = \frac{\kappa_5}{3} \bar{T}_{00} \quad (4.9)$$

with $\varepsilon = +1$. Furthermore, if matter is a perfect fluid, one can decompose the energy-momentum tensor into bulk and brane parts as [17]

$$\begin{aligned}\bar{T}_A^B(t, y) &= (\bar{T}_A^B)_{\text{bulk}} + (\bar{T}_A^B)_{\text{brane}} , \\ (\bar{T}_A^B)_{\text{brane}} &= \frac{\delta(y)}{\phi} \text{diag}(-\rho(t), P(t), P(t), P(t), 0) ,\end{aligned}\tag{4.10}$$

where $(\bar{T}_A^B)_{\text{bulk}}$ represents as the *vacuum energy-momentum tensor* or the *cosmological constant* $(\Lambda_5/\kappa_5)\eta_A^B$ in the bulk, and $\rho(t)$ and $P(t)$ are the energy density and the pressure of the normal matter localized on the brane, respectively.

For a discontinuous first derivative of the bulk metric \bar{g} ($\in C^1(M) \setminus \bigcup_{k>1} C^k(M)$), the Dirac delta function would appears in its second derivative, so that the FLRW metric could lead to the equation of the scale factor with the form at $y = 0$

$$a''(t, y) = \delta(y) [a'](t, 0) + \tilde{a}''(t, y) ,\tag{4.11}$$

where \tilde{a}'' denotes the non-distributional part of a'' and the definition of the *jump* is

$$[f](0) := \lim_{\delta \rightarrow 0^+} f(\delta) - f(-\delta) \quad (f : M \rightarrow \mathbb{R}) ,\tag{4.12}$$

which measures the discontinuity of a real-valued function f across the brane. With the form of the scale factor, (4.9) yields the junction condition

$$[a'](t, 0) = \frac{\kappa_5}{3\varepsilon} \rho a_0(t) \phi_0(t)\tag{4.13}$$

where $a_0(t) := a(t, 0)$ and $\phi_0(t) := \phi(t, 0)$ are considered as the scalar factor and a scalar field on the brane, respectively. Furthermore, if we impose the so-called \mathbb{Z}_2 symmetry [4] for the scale factor in Eq. (4.13) on the bulk as a real-valued quantity f must be an odd function $f(x) = -f(-x)$ across the brane, we obtain the Friedmann equation on the brane to be

$$\frac{\dot{a}_0^2(t)}{a_0^2(t)} + \frac{\ddot{a}_0(t)}{a_0(t)} = -\frac{\kappa_5^2}{36} \rho(t)(\rho(t) + 3P(t)) - \frac{k_5}{3\phi_0^2(t)} (\bar{T}_{55})_{\text{bulk}} ,\tag{4.14}$$

which is the same as the braneworld theory of GR shown in [17]. Hence, we confirm that the cosmological braneworld scenario in TEGR coincides with that of GR, *i.e.*, there is no distinguish between TEGR and GR in the braneworld FLRW cosmology, which again justifies the name of TEGR.

The physical consequence of the cosmological brane scenario here then follows from the discussions in [17]. In particular, if the extra 5th-dimension is compact, one can check if the solutions of $a(t, y)$ and $\phi(t, y)$ derived from (4.8) are well-defined ones, as given in [17].

Finally, we remark that the Friedmann equation (4.8) in the bulk can be identified as $G_{00} = -\kappa_5 \bar{T}_{00}$ and $G_{05} = 0$, which are the same as those in [17]. This result implies that a *radiating* contribution of the universe can be generated in TEGR due to the extra spatial dimension. It can be viewed as a generic property that there exists a component of *dark radiation* in the braneworld scenario. We have to mention that there is no extrinsic curvature in TEGR since the projected effects of the dark radiation and discontinuity property of the brane come from torsion itself, which is clearly beyond the expectations of GR [20, 22] as already pointed out in [13].

V. CONCLUSIONS

We have studied teleparallel gravity in five-dimensional spacetime. In particular, we have shown that the Kaluza-Klein theory in teleparallel gravity do not generate a Brans-Dicke type of the effective 4-dimensional Lagrangian as GR. This result is different from that given in [12]. We have demonstrated that the braneworld theory of teleparallel gravity in the FLRW cosmology provides an equivalent viewpoint as Einstein's general relativity. The additional radiation of the universe can arise from the extra dimension, which is a generic feature in the braneworld theory. In GR, the extrinsic curvature plays an important role to give the projected effects in the lower dimension, which indicates the embedding can lead to different dynamics of the hypersurface. On the other hand, in TEGR the projected effect on the lower dimensional manifold is determined by the projection of torsion 2-forms.

ACKNOWLEDGMENTS

We are grateful to Professor Friedrich W. Hehl for the encouragement. We would like to thank Keisuke Izumi, Yen-Chin Ong and Yi-Peng Wu for useful discussions. The work was supported in part by National Center for Theoretical Sciences, National Science Council (NSC-101-2112-M-007-006-MY3) and National Tsing-Hua University (102N2725E1), Taiwan, R.O.C.

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